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The UV and IR Origin of Nonabelian Chiral Gauge Anomalies on Noncommutative Minkowski Space-time

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We discuss both the UV and IR origin of the one-loop triangle gauge anomalies for noncommutative nonabelian chiral gauge theories with fundamental, adjoint and bi-fundamental fermions for $U(N)$ groups. We find that gauge anomalies only come from planar triangle diagrams, the non-planar triangle contributions giving rise to no breaking of the Ward identeties. Generally speaking, theories with fundamental and bi-fundamental chiral matter are anomalous. Theories with only adjoint chiral fermions are anomaly free.

1. Introduction

Let space-time be noncommutative [1] Minkowski and let ψ denote a fermion chirally coupled to a $U(N)$ gauge field A_μ. Let A_μ be an $N \times N$ matrix which transforms under an infinitesimal gauge transformation as follows

$$\big(\delta_\omega \mathrm{A}_\mu\big)^i{}_j = \partial_\mu \omega^i{}_j - i\, \mathrm{A}^i_{\mu\, k} \star \omega^k{}_j + i\, \omega^i{}_k \star \mathrm{A}^k_{\mu\, j}, \tag{1}$$

where $\omega^i{}_j = \omega^{*\, j}{}_i$, $i, j = 1, \cdots, N$, are the infinitesimal gauge transformation parameters and the symbol $\star$ stands for the Moyal product of functions on Minkowski space-time. The Moyal product is defined thus

$$(f \star g)(x) = e^{\frac{i}{2}\theta^{\mu\nu}\partial^u_\mu \partial^w_\nu} f(u) g(w)\, |_{u=x, w=x},$$

where $\theta^{\mu\nu}$ is an antisymmetric real matrix either of magnetic type or light-like type [2].

Following ref. [3], we introduce three basic right-handed chiral gauge transformation laws for the fermion field

$$(\delta_\omega \psi)^i = i\, \omega^i{}_j \star \mathrm{P}_+\, \psi^j \qquad \text{and} \qquad (\delta_\omega \bar\psi)_k = -i \bar\psi_k \star \omega^k_i \mathrm{P}_-, \tag{2}$$

$$(\delta_\omega \psi)_j = -i\, \mathrm{P}_+\, \psi_i \star \omega^i{}_j \qquad \text{and} \qquad (\delta_\omega \bar\psi)^k = i \omega^k_i \star \bar\psi^i\, \mathrm{P}_- \tag{3}$$

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and

$$(\delta_\omega\psi)^i_{\ j} = i\left(\omega^i_{\ j}\star \mathrm{P}_+\,\psi^j_{\ i} - \mathrm{P}_+\,\psi^i_{\ j}\star\omega^j_{\ i}\right) \qquad \text{and} \qquad (\delta_\omega\bar\psi)^k_{\ i} = -i\left(\bar\psi^k_{\ j}\star\omega^j_{\ i}\,\mathrm{P}_- - \omega^k_{\ j}\star\bar\psi^j_{\ i}\,\mathrm{P}_-\right). \tag{4}$$

As usual, $\mathrm{P}_+ = \frac{1}{2}(1+\gamma_5)$. The fermions transforming under gauge transformations as in eqs. (2), (3) and (4) will be called (right-handed) fundamental, (right-handed) anti-fundamental and (right-handed) adjoint fermions, respectively.

The $U(N)$ chiral gauge theories with the fermion ψ transforming as in eqs. (2), (3) and (4) are governed, respectively, by the following classical actions

$$\mathrm{S} \;=\; \int \mathrm{d}^4x\;\bar\psi_i\star(i\partial\!\!\!/\psi^i + \mathrm{A}^i_{\mu\,j}\star\gamma^\mu \mathrm{P}_+\psi^j), \tag{5}$$

$$\mathrm{S} \;=\; \int \mathrm{d}^4x\;\bar\psi^i\star(i\partial\!\!\!/\psi_i - \gamma_\mu \mathrm{P}_+\psi_j\star\mathrm{A}^j_{\mu\,i}), \tag{6}$$

and

$$\mathrm{S} \;=\; \int \mathrm{d}^4x\;\bar\psi^k_{\ i}\star(i\partial\!\!\!/\psi^i_{\ k} + \mathrm{A}^i_{\mu\,j}\star\gamma^\mu\mathrm{P}_+\psi^j_{\ k} - \gamma^\mu\mathrm{P}_+\psi^i_{\ j}\star\mathrm{A}^j_{\mu\,k}). \tag{7}$$

Each action is invariant under the corresponding chiral gauge transformations; these transformations are displayed in eqs. (1), (2), (3) and (4).

The effective action, $\Gamma[\mathrm{A}]$, which arises upon integrating out the fermionic degrees of freedom is formally given by

$$e^{i\Gamma[\mathrm{A}]} \;=\; \int \mathrm{d}\psi d\bar\psi\; e^{iS[A,\psi,\bar\psi]}, \tag{8}$$

with $S[A,\psi,\bar\psi]$ given by any of the classical actions in eqs. (5), (6) and (7). The path integral above is formally invariant under the corresponding chiral gauge transformations -see eqs. (1), (2), (3) and (4), which leads, formally, to the gauge invariance of $\Gamma[\mathrm{A}]$. And yet, it has been shown in ref. [4] that once the path integral is properly defined *á la* Berezin the effective action is no longer gauge invariant for fermions transforming as in eqs. (2), (3), but rather the following anomaly equation holds

$$\delta_\theta\Gamma[A] \;=\; \pm\frac{1}{24\pi^2}\,\mathrm{Tr}\int d^4x\,\varepsilon^{\mu_1\mu_2\mu_3\mu_4}\,\theta\,\partial_{\mu_1}\big[\mathrm{A}_{\mu_2}\star\partial_{\mu_3}\mathrm{A}_{\mu_4} - \frac{i}{2}\mathrm{A}_{\mu_2}\star\mathrm{A}_{\mu_3}\star\mathrm{A}_{\mu_4}\big]. \tag{9}$$

Where the overall $+$ and $-$ signs are for right-handed fundamental and right-handed anti-fundamental fermions, respectively. This equation can also be obtained by using standard diagrammatic techniques. One begins by working out the anomaly equation for the three-point contribution -the famous triangle diagrams- to $\Gamma[\mathrm{A}]$, the latter has been defined in eq. (8), and, then, one uses the Wess-Zumino consistency condition [5] to obtain the complete equation. Agreement with eq. (9) demands that this triangle anomaly reads

$$\begin{aligned} &p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps} = \mp\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta \\ &\Big(\mathrm{Tr}\,\{\mathrm{T}^{a_1},\mathrm{T}^{a_2}\}\,\mathrm{T}^{a_3}\cos\frac{1}{2}\theta(p_1,p_2) - i\mathrm{Tr}\,[\mathrm{T}^{a_1},\mathrm{T}^{a_2}]\mathrm{T}^{a_3}\sin\frac{1}{2}\theta(p_1,p_2)\Big), \end{aligned} \tag{10}$$

where $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ gives the Fourier transform,

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)=(2\pi)^4\delta(p_1+p_2+p_3)\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2),$$

of the three-point function

$$\frac{\delta^3 i\Gamma[\mathrm{A}]}{\delta\mathrm{A}^{a_1}_{\mu_1}(x_1)\delta\mathrm{A}^{a_2}_{\mu_2}(x_2)\delta\mathrm{A}^{a_3}_{\mu_3}(x_3)}\Bigg|_{A=0}=\int\prod_{i=1}^{3}\frac{d^4p_i}{(2\pi)^4}\,e^{ip_ix}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3),\tag{11}$$

and the superscript eps stands for the contribution to this Green function which carries the Levi-Civita pseudotensor. The indices a_1,a_2 and a_3 run over the generators of the gauge group. The symbol $\theta(p_1,p_2)$ is a shorthand for $p_{1\,\mu}\,\theta^{\mu\nu}\,p_{2\,\nu}$. Eq. (10) leads clearly to the conclusion that the triangle contribution on noncommutative Minkowski for (anti-) fundamental chiral fermions is anomaly free if, and only if,

$$\mathrm{Tr}\,\mathrm{T}^{a_1}\mathrm{T}^{a_2}\mathrm{T}^{a_3}=0;$$

its ordinary counterpart being $\mathrm{Tr}\,\{\mathrm{T}^{a_1},\mathrm{T}^{a_2}\}\mathrm{T}^{a_3}=0$.

We can also have chiral gauge theories with bi-fundamental chiral fermions $\psi^{i}_{R\,j}=\mathrm{P}_+\psi^i_{\;j}$, $i=1,\cdots,N$ and $j=1,\cdots,M$ [3]. Now the fermion couples to a $U(N)$ gauge field, say, A_μ and a $U(M)$ gauge field, say, B_μ, the former being an $N\times N$ matrix and the latter an $M\times M$ matrix. The classical action for this theory reads

$$\mathrm{S}\;=\;\int\mathrm{d}^4x\;\bar\psi^{k}_{\;i}\star(i\partial\!\!\!/\psi^i_{\;k}+\mathrm{A}^{i}_{\mu\,j}\star\gamma^\mu\mathrm{P}_+\psi^j_{\;k}-\gamma^\mu\mathrm{P}_+\psi^i_{\;j}\star\mathrm{B}^{j}_{\mu\,k}),\tag{12}$$

This action is invariant under the following infinitesimal gauge transformations

$$\begin{aligned}
(\delta_{(\omega,\chi)}\psi)^i_{\;j}&=i\Big(\omega^i_{\;j}\star\mathrm{P}_+\psi^j_{\;i}-\mathrm{P}_+\psi^i_{\;j}\star\chi^j_{\;i}\Big),\\
(\delta_{(\omega,\chi)}\bar\psi)^k_{\;i}&=-i\Big(\bar\psi^k_{\;j}\star\omega^j_{\;i}\mathrm{P}_--\chi^k_{\;j}\star\bar\psi^j_{\;i}\mathrm{P}_-\Big),\\
\big(\delta_\omega\mathrm{A}_\mu\big)^i_{\;j}&=\partial_\mu\omega^i_{\;j}-i\,\mathrm{A}^i_{\mu\,k}\star\omega^k_{\;j}+i\,\omega^i_{\;k}\star\mathrm{A}^k_{\mu\,j},\\
\big(\delta_\chi\mathrm{B}_\mu\big)^i_{\;j}&=\partial_\mu\chi^i_{\;j}-i\,\mathrm{B}^i_{\mu\,k}\star\chi^k_{\;j}+i\,\chi^i_{\;k}\star\mathrm{B}^k_{\mu\,j},
\end{aligned}$$

where $\omega^i_{\;j}=\omega^{*\,j}_{\;\;\;i}$, $i,j=1,\cdots,N$, and $\chi^i_{\;j}=\chi^{*\,j}_{\;\;\;i}$, $i,j=1,\cdots,M$, are the infinitesimal gauge transformation parameters.

The effective action, $\Gamma[\mathrm{A},\mathrm{B}]$, that one obtains by integrating over the fermionic degrees of freedom formally reads thus

$$e^{i\Gamma[\mathrm{A},\mathrm{B}]}\;=\;\int\mathrm{d}\psi d\bar\psi\;e^{iS[\mathrm{A},\mathrm{B},\psi,\bar\psi]},\tag{13}$$

with $S[\mathrm{A},\mathrm{B},\psi,\bar\psi]$ given in eq. (12). We shall see that in general there are triangle gauge anomalies jeopardizing the formal gauge invariance of $\Gamma[\mathrm{A},\mathrm{B}]$.

It is well known that the chiral gauge anomaly on ordinary Minkowski space-time can be understood either as a short-distance phenomenon (UV effect) [6] or as an IR effect (large-distance phenomenon) [7]. The purpose of this paper is to show that nonabelian chiral anomalies on noncommutative Minkowski space-time can also be explained either as an UV effect or an IR phenomenon. Recall that if the chiral fermions of the theory are either adjoint or bi-fundamental, there are non-planar contributions to the three-point function of the effective action ($\Gamma_{\rm adj}[{\rm A}]$ and $\Gamma[{\rm A},{\rm B}]$ in eqs. (8) and (13)) and one wonders whether these non-planar contributions may give rise to some gauge anomaly due to its noncommutative IR structure; this structure being a consequence of their being regularized in the UV by the appropriate Moyal exponentials [8]. We shall show in this paper that, at least for the theories we have studied, there are no anomalous contributions coming from the nonplanar triangle diagrams: gauge anomalies -if they exist- are due to planar triangle diagrams. We have assumed that, as in the ordinary case, true anomalies always involve the Levy-Civita pseudotensor. Standard arguments [9] can put forward to support this assumption.

The layout of this paper is as follows. Section 2 is devoted the analysis of the of the anomaly equation -eq. (10)- as an UV effect. In this section we will also show that the chiral gauge theory whose classical action given in eq. (7) is anomaly free. We close the section by computing the triangle gauge anomalies for a chiral theory with a bi-fundamental right-handed fermion and conclude that they only come from the planar contribution to its effective action; the nonplanar part being thus anomaly free. In section 3 we shall exhibit the IR origin of the nonabelian chiral anomalies we have worked out in section 2. We include an Appendix with the relevant Feynman integrals.

2. The UV origin of nonabelian chiral gauge anomalies

Let us begin with the chiral theory whose action is given by eq. (5). The UV character of eq. (10) is made apparent by computing its l.h.s with the help of a regularization method. We shall use dimensional regularization as defined by Breitenlohner and Maison [10] (see ref. [11] for an alternative), i.e., with the definition of γ_5 given by 't Hooft and Veltman, and take the following classical action in the "d-dimensional" space of dimensional regularization (see ref. [12] and references therein):

$$\mathrm{S} \;=\; \int \mathrm{d}^d x\; \bar{\psi}_i \star (i\partial\!\!\!/\psi^i + A^a_\mu T^{a\,i}{}_j \bar{\gamma}^\mu \mathrm{P}_+ \star \psi^j).$$

Here, $T^{a\,i}{}_j \;=\; T^{*\,a\,j}{}_i$. The object denoted by the symbol $\bar{\gamma}^\mu$ and the other objects in the algebra of "d-dimensional" covariants are defined as in section 2. of ref. [12]. The "d-dimensional" counterpart of $\theta^{\mu\nu}$ is defined as an object which satisfies

$$\theta^{\mu\nu} = -\theta^{\nu\mu}, \quad \hat{g}_{\mu\rho}\theta^{\rho\nu} = 0, \quad p_\mu\, \theta^{\mu\rho}\eta_{\rho\sigma}\theta^{\sigma\nu}\, p_\nu \geq 0,\; \forall p_\mu.$$

The Feynman rules needed to reproduce our computations are given in figure 1.

Let us define the dimensionally regularized counterpart of the l.h.s of eq. (10):

$$\triangle^{a_1 a_2 a_3}_{\mu_1\mu_2}(p_1,p_2;d) = p_3^{\mu_3}\, \Gamma^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2;d)^{\rm eps}.$$

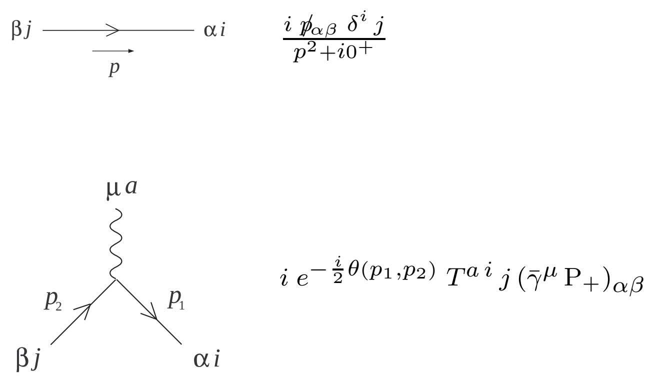

Figure 1.

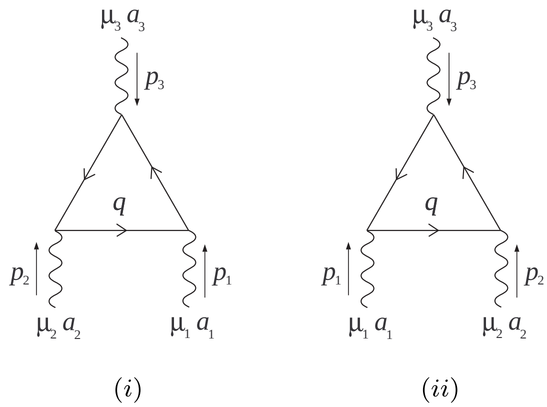

Figure 2.

At the one-loop level $\triangle^{a_1a_2a_3}_{\mu_1\mu_2}(p_1,p_2;d)$ is given by the sum of the contributions coming from the two triangle diagrams in figure 2. This sum reads

$$\begin{aligned}\triangle^{a_1a_2a_3}_{\mu_1\mu_2}(p_1,p_2;d) =&e^{-\frac{i}{2}\theta(p_1,p_2)}\,\mathrm{Tr}\,\mathrm{T}^{a_1}\mathrm{T}^{a_2}\mathrm{T}^{a_3}\,\triangle^{(1)}_{\mu_1\mu_2}(p_1,p_2;d)+\\ &e^{\frac{i}{2}\theta(p_1,p_2)}\,\mathrm{Tr}\,\mathrm{T}^{a_2}\mathrm{T}^{a_1}\mathrm{T}^{a_3}\,\triangle^{(2)}_{\mu_1\mu_2}(p_1,p_2;d),\end{aligned} \tag{14}$$

with

$$\triangle^{(1)}_{\mu_1\mu_2}(p_1,p_2;d) = -\int\frac{d^dq}{(2\pi)^d}\frac{\mathrm{tr}^{\,\mathrm{eps}}\{(\not{q}+\not{p}_1)\,\bar{\gamma}_{\mu_1}\mathrm{P}_+\,\not{q}\bar{\gamma}_{\mu_2}\mathrm{P}_+\,(\not{q}-\not{p}_2)(\bar{\not{p}}_1+\bar{\not{p}}_2)\mathrm{P}_+\}}{(q^2+i0^+)((q+p_1)^2+i0^+)((q-p_2)^2+i0^+)},$$

and

$$\triangle^{(2)}_{\mu_1\mu_2}(p_1,p_2;d) = -\int\frac{d^dq}{(2\pi)^d}\frac{\mathrm{tr}^{\,\mathrm{eps}}\{(\not{q}+\not{p}_2)\,\bar{\gamma}_{\mu_2}\mathrm{P}_+\,\not{q}\bar{\gamma}_{\mu_1}\mathrm{P}_+\,(\not{q}-\not{p}_1)(\bar{\not{p}}_1+\bar{\not{p}}_2)\mathrm{P}_+\}}{(q^2+i0^+)((q+p_2)^2+i0^+)((q-p_1)^2+i0^+)}. \tag{15}$$

In the previous equation $\mathrm{tr}^{\,\mathrm{eps}}$ shows that only contributions involving the Levi-Civita symbol $\varepsilon_{\mu_1\mu_2\mu_3\mu_4}$ are kept upon computing the trace over the gammas.

Now, the Feynman diagrams in figure 2 are planar; hence, it can be readily seen [13] that their noncomutative character is completely embodied (see eq. (14)) in the overall phase factors $e^{-\frac{i}{2}\theta(p_1,p_2)}$ and $e^{\frac{i}{2}\theta(p_1,p_2)}$. Then, it does not come as a surprise that equation (10) holds, for the Feynman integrals in eq. (15) are the standard integrals whose UV behaviour give rise to the nonabelian chiral anomaly on commutative Minkowski space.

Taking into account that $\mathrm{P}_+\hat{\gamma}_\mu\bar{\gamma}_\nu\mathrm{P}_+=0$ and performing some standard manipulations one shows that

$$
\begin{aligned}
&\triangle^{(1)}_{\mu_1\mu_2}(p_1,p_2;d) \;= \\
&-\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\frac{\mathrm{tr}\big\{\bar{\gamma}_\alpha\bar{\gamma}_{\mu_1}\bar{\gamma}_\beta\bar{\gamma}_{\mu_2}\gamma_5\big\}\,\bar{p}_1^\alpha(\bar{q}+\bar{p}_2)^\beta\,\hat{q}^2}{(q^2+i0^+)((q+p_2)^2+i0^+)((q+p_1+p_2)^2+i0^+)}\\
&-\frac{1}{2}\int\frac{d^dq}{(2\pi)^d}\frac{\mathrm{tr}\big\{\bar{\gamma}_{\mu_1}\bar{\gamma}_\alpha\bar{\gamma}_{\mu_2}\bar{\gamma}_\beta\gamma_5\big\}\,(\bar{q}+\bar{p}_1)^\alpha\bar{p}_2^\beta\,\hat{q}^2}{(q^2+i0^+)((q+p_1)^2+i0^+)((q+p_1+p_2)^2+i0^+)}.
\end{aligned}
\tag{16}
$$

Notice that the integrand of the integrals in eq. (16) formally vanishes in the limit $d\to 4$, since it contains the evanescent term $\hat{q}^2$. However, the limit $d\to 4$ of these integrals although finite is not zero. Indeed, if we take into account that $\hat{q}^2=q^\alpha q^\beta\hat{g}_{\alpha\beta}$, we readily see that what we are facing is the computation of integrals which are UV divergent by power-counting at $d=4$ and which will develop a simple pole at $d=4$ when computed in dimensional regularization (notice that the integrals at hand are IR finite by power-counting at nonexceptional momenta). This pole will be canceled at the end of the day by the evanescent (order $d-4$) contribution coming from the contraction with $\hat{g}_{\alpha\beta}$, yielding a polynomial in the external momenta (short distance operator) as value for $\triangle^{(1)}_{\mu_1\mu_2}(p_1,p_2;d)$ at $d=4$. We have thus explained the nonabelian chiral anomaly of eq. (10) as an UV effect. Indeed, a little computation shows that the integrals in eq. (16) yield the following result

$$
\triangle^{(1)}_{\mu_1\mu_2}(p_1,p_2;d=4)\;=\;-\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta. \tag{17}
$$

The reader may find the following integrals useful

$$
\begin{aligned}
\int\frac{d^dq}{(2\pi)^d}\frac{\hat{q}^2}{q^2\,(q+p_2)^2\,(q+p_1+p_2)^2} &= -\frac{i}{16\pi^2}\big(\frac{1}{2}\big)\;+\;O(d-4),\\
\int\frac{d^dq}{(2\pi)^d}\frac{\hat{q}^2\,\bar{q}^\alpha}{q^2\,(q+p_2)^2\,(q+p_1+p_2)^2} &= \frac{i}{16\pi^2}\big(\frac{1}{6}\big)(\bar{p}_1+2\bar{p}_2)^\alpha\;+\;O(d-4).
\end{aligned}
$$

It is clear that for $\triangle^{(2)}_{\mu_1\mu_2}(p_1,p_2;d)$ in eq. (15) one will obtain the following finite answer

$$
\triangle^{(2)}_{\mu_1\mu_2}(p_1,p_2;d=4)\;=\;-\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta. \tag{18}
$$

Finally, if we substitute this result and eq. (17) in eq. (14), we will recover the one-loop triangle anomaly of eq. (10).

A completely similar analysis can be done for the chiral theory defined by the action in eq. (6). Let us move on and compute $p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\mathrm{eps}}$ for the theory with adjoint

fermionic matter. The classical action of this theory is given in eq. (7). The Ward identity that should hold if the gauge symmetry of the classical theory is a symmetry of the quantum theory reads

$$\int d^4x\,\omega^{i_1}_{\ i_2}\star\partial_\mu\frac{\delta\Gamma[\mathrm{A}]}{\delta\mathrm{A}^{i_1}_{\mu\,i_2}}\;=\;i\int d^4x\,\omega^{i_1}_{\ i_2}\star\left[\mathrm{A}^{i_2}_{\mu\,i_3}\star\frac{\delta\Gamma[\mathrm{A}]}{\delta\mathrm{A}^{i_1}_{\mu\,i_3}}-\frac{\delta\Gamma[\mathrm{A}]}{\delta\mathrm{A}^{i_3}_{\mu\,i_2}}\star\mathrm{A}^{i_3}_{\mu\,i_1}\right].$$

For $p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}$, the previous equation boils down to

$$p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}=0.$$

To obtain this equation it is necessary to take into account that the two-point contribution to $\Gamma[\mathrm{A}]$ has no pseudotensor contribution.

The dimensional regularization counterpart of the action in eq. (7) will have for us the following expression

$$\mathrm{S}\;=\;\int\mathrm{d}^dx\;\bar\psi^k_{\ i}\star(i\partial\!\!\!/\psi^i_{\ k}+\mathrm{A}^i_{\mu\,j}\star\bar\gamma^\mu\mathrm{P}_+\psi^j_{\ k}-\bar\gamma^\mu\mathrm{P}_+\psi^i_{\ j}\star\mathrm{A}^j_{\mu\,k}),$$

with the same notation as at the beginning of this section. Instead of deriving Feynman rules from this action and compute $p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}$ from the corresponding triangle diagrams in fig. 2, we shall follow an alternative procedure which will supply a more thorough understanding of the final answer. Let us introduce first the following chiral current in the "d-dimensional" space of dimensional regularization

$$j^{\,a}_{\ \mu}(x)\equiv i\frac{\delta S[\mathrm{A}]}{\mathrm{A}^a_\mu(x)}\equiv j^{\,a\,-}_{\ \mu}(x)+j^{\,a\,+}_{\ \mu}(x),\tag{19}$$

where

$$j^{\,a\,-}_{\ \mu}(x)=-i\,\psi^j_{\ k\,\beta}\star\bar\psi^k_{\ i\,\alpha}(x)\,\mathrm{T}^{a\,i}_{\ \ \ j}\left(\bar\gamma^\mu\mathrm{P}_+\right)_{\alpha\beta}$$

and

$$j^{\,a\,+}_{\ \mu}(x)=-i\,\bar\psi^k_{\ i\,\alpha}\star\psi^i_{\ j\,\beta}(x)\mathrm{T}^{a\,j}_{\ \ \ k}\left(\bar\gamma^\mu\mathrm{P}_+\right)_{\alpha\beta}.\tag{20}$$

Let $j^{\,a\,(\cdot)}_{\ \mu}(p)$ be given by

$$j^{\,a\,(\cdot)}_{\ \mu}(x)\;=\;\int\frac{d^4p}{(4\pi)^4}\;e^{ipx}\;j^{\,a\,(\cdot)}_{\ \mu}(p).$$

Then, the three-point function (eq. (11)) in momentum space reads

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)=\langle j^{\,a_1}_{\ \ \mu_1}(p_1)\,j^{\,a_2}_{\ \ \mu_2}(p_2)\,j^{\,a_3}_{\ \ \mu_3}(p_3)\rangle_{\rm con}.$$

Where the subscript "con" refers to the connected part of the corresponding Green function. Throughout this paper, vacuum expectations values are computed with the free fermionic action. Taking into account eq. (19), we obtain

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)\;=\;\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)_{\rm P}\;+\;\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)_{\rm NP},$$

where

$$\Gamma^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)_{\rm P} = \langle j^{a_1}{}_{\mu_1}^{-}(p_1)\, j^{a_2}{}_{\mu_2}^{-}(p_2)\, j^{a_3}{}_{\mu_3}^{-}(p_3)\rangle_{\rm con} + \langle j^{a_1}{}_{\mu_1}^{+}(p_1)\, j^{a_2}{}_{\mu_2}^{+}(p_2)\, j^{a_3}{}_{\mu_3}^{+}(p_3)\rangle_{\rm con} \tag{21}$$

and

$$\begin{aligned}\Gamma^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)_{\rm NP} = &\langle j^{a_1}{}_{\mu_1}^{-}(p_1) j^{a_2}{}_{\mu_2}^{-}(p_2) j^{a_3}{}_{\mu_3}^{+}(p_3)\rangle_{\rm con} + \langle j^{a_1}{}_{\mu_1}^{+}(p_1) j^{a_2}{}_{\mu_2}^{+}(p_2) j^{a_3}{}_{\mu_3}^{-}(p_3)\rangle_{\rm con}\\ &\langle j^{a_1}{}_{\mu_1}^{-}(p_1) j^{a_2}{}_{\mu_2}^{+}(p_2) j^{a_3}{}_{\mu_3}^{-}(p_3)\rangle_{\rm con} + \langle j^{a_1}{}_{\mu_1}^{+}(p_1) j^{a_2}{}_{\mu_2}^{-}(p_2) j^{a_3}{}_{\mu_3}^{+}(p_3)\rangle_{\rm con}\\ &\langle j^{a_1}{}_{\mu_1}^{+}(p_1) j^{a_2}{}_{\mu_2}^{-}(p_2) j^{a_3}{}_{\mu_3}^{-}(p_3)\rangle_{\rm con} + \langle j^{a_1}{}_{\mu_1}^{-}(p_1) j^{a_2}{}_{\mu_2}^{+}(p_2) j^{a_3}{}_{\mu_3}^{+}(p_3)\rangle_{\rm con}.\end{aligned} \tag{22}$$

The subscripts "P" and "NP" refer, respectively, to the planar and nonplanar parts of $\Gamma^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)$. The reader may easily realize that only when the three currents in the correlation function carry the same superscript, $-$ or $+$, there is no Moyal exponential carrying the loop momenta. Notice that each correlation function of the type $\langle j\,j\,j\rangle^{\rm con}$ above can be interpreted as the sum of two triangle diagrams with vertices given by the currents of the former.

Now, taking into account eq. (20), it can be easily shown that the the Green functions contributing to the r.h.s of eq. (21) satisfy

$$\begin{aligned}&p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}^{-}(p_1)\, j^{a_2}{}_{\mu_2}^{-}(p_2)\, j^{a_3}{}_{\mu_3}^{-}(p_3)\rangle^{\rm eps}_{\rm con} = (2\pi)^4\delta(p_1+p_2+p_3)\,N\\ &e^{-\frac{i}{2}\theta(p_1,p_2)}\,{\rm Tr}\,{\rm T}^{a_1}{\rm T}^{a_2}{\rm T}^{a_3}\,\triangle^{(1)}_{\mu_1\mu_2}(p_1,p_2;d) + e^{\frac{i}{2}\theta(p_1,p_2)}\,{\rm Tr}\,{\rm T}^{a_2}{\rm T}^{a_1}{\rm T}^{a_3}\,\triangle^{(2)}_{\mu_1\mu_2}(p_1,p_2;d),\\ &p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}^{+}(p_1)\, j^{a_2}{}_{\mu_2}^{+}(p_2)\, j^{a_3}{}_{\mu_3}^{+}(p_3)\rangle^{\rm eps}_{\rm con} = -(2\pi)^4\delta(p_1+p_2+p_3)\,N\\ &e^{-\frac{i}{2}\theta(p_1,p_2)}\,{\rm Tr}\,{\rm T}^{a_1}{\rm T}^{a_2}{\rm T}^{a_3}\,\triangle^{(2)}_{\mu_1\mu_2}(p_1,p_2;d) + e^{\frac{i}{2}\theta(p_1,p_2)}\,{\rm Tr}\,{\rm T}^{a_2}{\rm T}^{a_1}{\rm T}^{a_3}\,\triangle^{(1)}_{\mu_1\mu_2}(p_1,p_2;d),\end{aligned} \tag{23}$$

where $\triangle^{(1)}_{\mu_1\mu_2}(p_1,p_2;d)$ and $\triangle^{(2)}_{\mu_1\mu_2}(p_1,p_2;d)$ are given in eq. (15) and the superscript "eps" indicates that one should keep only contributions involving the Levi-Civita symbol. Now, substituting eqs. (17) and (18) in eq. (23), one obtains that the following equations hold at $d=4$

$$\begin{aligned}&p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}^{-}(p_1)\, j^{a_2}{}_{\mu_2}^{-}(p_2)\, j^{a_3}{}_{\mu_3}^{-}(p_3)\rangle^{\rm eps}_{\rm con} = -(2\pi)^4\delta(p_1+p_2+p_3)\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta\\ &\qquad N\Big(\,{\rm Tr}\,\{{\rm T}^{a_1},{\rm T}^{a_2}\}\,{\rm T}^{a_3}\cos\frac{1}{2}\theta(p_1,p_2) - i{\rm Tr}\,[{\rm T}^{a_1},{\rm T}^{a_2}]{\rm T}^{a_3}\sin\frac{1}{2}\theta(p_1,p_2)\Big),\\ &p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}^{+}(p_1)\, j^{a_2}{}_{\mu_2}^{+}(p_2)\, j^{a_3}{}_{\mu_3}^{+}(p_3)\rangle^{\rm eps}_{\rm con} = (2\pi)^4\delta(p_1+p_2+p_3)\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta\\ &\qquad N\Big(\,{\rm Tr}\,\{{\rm T}^{a_1},{\rm T}^{a_2}\}\,{\rm T}^{a_3}\cos\frac{1}{2}\theta(p_1,p_2) + i{\rm Tr}\,[{\rm T}^{a_1},{\rm T}^{a_2}]{\rm T}^{a_3}\sin\frac{1}{2}\theta(p_1,p_2)\Big).\end{aligned}$$

Hence, each correlation function of currents contributing to the r.h.s of eq. (21) yields an anomalous term, but its sum, i.e., the planar part of $\Gamma^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)^{\rm eps}$, carries no anomaly:

$$p_3^{\mu_3}\,\Gamma^{a_1 a_2 a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)^{\rm eps}_{\rm P} \;=\; 0.$$

The reader should notice that result we have just derived can be understood as follows: the sum of the two triangle diagrams contributing to $\langle j^{a_1}{}_{\mu_1}^{-}(p_1)\, j^{a_2}{}_{\mu_2}^{-}(p_2)\, j^{a_3}{}_{\mu_3}^{-}(p_3)\rangle_{\rm con}$ yield a chiral anomaly opposite to the chiral anomaly coming from the sum of the two triangle diagrams contributing to $\langle j^{a_1}{}_{\mu_1}^{+}(p_1)\, j^{a_2}{}_{\mu_2}^{+}(p_2)\, j^{a_3}{}_{\mu_3}^{+}(p_3)\rangle_{\rm con}$, i.e., the contribution coming from the fermionic modes in the fundamental representation of $U(N)$ moving around the loop cancels the contribution furnished by the fermionic modes in the anti-fundamental representation of $U(N)$ propagating along the loop: recall that the adjoint representation of $U(N)$ can be understood as the product of its fundamental and anti-fundamental representations.

Let us now show that there is no anomaly in the pseudotensor part of the nonplanar contribution given in eq. (22). Here, of course, we shall meet only integrals which give UV finite results at $d=4$ -since the Moyal exponential regulate them in the UV-, but which develop, as a consequence, of the UV/IR connection in noncommutative field theories, IR divergences as one approaches the noncommutative IR region $\tilde{p}=0$. Let us see whether or not they carry any anomaly. For the first three-current correlation function on the r.h.s of eq. (22), one obtains the following intermediate results at $d=4$

$$p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}^{-}(p_1)\, j^{a_2}{}_{\mu_2}^{-}(p_2)\, j^{a_3}{}_{\mu_3}^{+}(p_3)\rangle_{\rm con}^{\rm eps} = (2\pi)^4\delta(p_1+p_2+p_3){\rm Tr}({\rm T}^{a_1}{\rm T}^{a_2}){\rm Tr}\,{\rm T}^{a_3}$$
$$\Big[e^{\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)+e^{-\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)\Big], \qquad (24)$$

with

$$\triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3) \;=\; \int\frac{d^4q}{(2\pi)^4}\,e^{-i\theta(q,p_3)}\,\frac{{\rm tr}^{\,\rm eps}\{(\not q+\not p_1)\,\gamma_{\mu_1}{\rm P}_+\not q\gamma_{\mu_2}{\rm P}_+(\not q-\not p_2)(\not p_1+\not p_2){\rm P}_+\}}{(q^2+i0^+)((q+p_1)^2+i0^+)((q-p_2)^2+i0^+)},$$

and

$$\triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3) \;=\; \int\frac{d^4q}{(2\pi)^4}\,e^{-i\theta(q,p_3)}\,\frac{{\rm tr}^{\,\rm eps}\{(\not q+\not p_2)\,\gamma_{\mu_2}{\rm P}_+\not q\gamma_{\mu_1}{\rm P}_+(\not q-\not p_1)(\not p_1+\not p_2){\rm P}_+\}}{(q^2+i0^+)((q+p_2)^2+i0^+)((q-p_1)^2+i0^+)}.$$

In the previous integrals $p_1+p_2+p_3=0$. Notice the characteristic Moyal factor, $e^{-i\theta(q,p_3)}$, of a nonplanar contribution. The integrals are well-defined provided we are off the noncommutative IR region defined by $\tilde{p}_3^2=0$. Let us show now that

$$e^{\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)+e^{-\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)=0. \qquad (25)$$

If we change variables $q\rightarrow q+p_2$ and $q\rightarrow q+p_1$ in $\triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)$ and $\triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)$, respectively, and use the cyclicity of the trace, we obtain

$$\begin{aligned}
&e^{\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)+e^{-\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)=\\
&e^{-\frac{i}{2}\theta(p_1,p_2)}\int\frac{d^4q}{(2\pi)^4}\,e^{-i\theta(q,p_3)}\,{\rm tr}^{\,\rm eps}\{\frac{1}{\not q}(\not p_1+\not p_2){\rm P}_+\frac{1}{\not q+\not p_1+\not p_2}\gamma_{\mu_1}{\rm P}_+\frac{1}{\not q+\not p_2}\gamma_{\mu_2}{\rm P}_+\}+\\
&e^{\frac{i}{2}\theta(p_1,p_2)}\int\frac{d^4q}{(2\pi)^4}\,e^{-i\theta(q,p_3)}\,{\rm tr}^{\,\rm eps}\{\frac{1}{\not q}(\not p_1+\not p_2){\rm P}_+\frac{1}{\not q+\not p_1+\not p_2}\gamma_{\mu_2}{\rm P}_+\frac{1}{\not q+\not p_1}\gamma_{\mu_1}{\rm P}_+\}.
\end{aligned} \qquad (26)$$

Now, using the equations $\gamma_\mu\gamma_\nu \mathrm{P}_+ = \mathrm{P}_+\gamma_\mu\gamma_\nu$, $\mathrm{P}_+^2 = \mathrm{P}_+$ and

$$(\not p_1 + \not p_2)\gamma_5 = -\not q\gamma_5 - \gamma_5(\not q + \not p_1 + \not p_2),$$

one readily casts the r.h.s of eq. (26) into the form

$$\begin{aligned}
&-\frac{1}{2}\, e^{-\frac{i}{2}\theta(p_1,p_2)} \int \frac{d^4q}{(2\pi)^4}\, e^{-i\theta(q,p_3)} \Big[\mathrm{tr}\big\{\gamma_5\, \frac{1}{\not q + \not p_1 + \not p_2}\, \gamma_{\mu_1}\, \frac{1}{\not q + \not p_2}\, \gamma_{\mu_2}\big\} + \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{tr}\big\{\frac{1}{\not q}\,\gamma_5\gamma_{\mu_1}\, \frac{1}{\not q + \not p_2}\, \gamma_{\mu_2}\big\}\Big] \\
&-\frac{1}{2} e^{\frac{i}{2}\theta(p_1,p_2)} \int \frac{d^4q}{(2\pi)^4}\, e^{-i\theta(q,p_3)} \Big[\mathrm{tr}\big\{\gamma_5\, \frac{1}{\not q + \not p_1 + \not p_2}\, \gamma_{\mu_2}\, \frac{1}{\not q + \not p_1}\, \gamma_{\mu_1}\big\} + \\
&\qquad\qquad\qquad\qquad\qquad\qquad \mathrm{tr}\big\{\frac{1}{\not q}\,\gamma_5\gamma_{\mu_2}\, \frac{1}{\not q + \not p_1}\, \gamma_{\mu_1}\big\}\Big].
\end{aligned} \tag{27}$$

Some Dirac algebra leads, respectively, to the following expressions

$$\begin{aligned}
\mathrm{tr}\big\{\gamma_5\, \frac{1}{\not q + \not p_1 + \not p_2}\, \gamma_{\mu_1}\, \frac{1}{\not q + \not p_2}\, \gamma_{\mu_2}\big\} &= -\mathrm{tr}\big\{\frac{1}{\not q + \not p_2}\, \gamma_5\gamma_{\mu_2}\, \frac{1}{\not q + \not p_1 + \not p_2}\, \gamma_{\mu_1}\big\}, \\
\mathrm{tr}\big\{\gamma_5 \frac{1}{\not q + \not p_1 + \not p_2}\, \gamma_{\mu_2}\, \frac{1}{\not q + \not p_1}\, \gamma_{\mu_1}\big\} &= -\mathrm{tr}\big\{\frac{1}{\not q + \not p_1}\, \gamma_5\gamma_{\mu_1} \frac{1}{\not q + \not p_1 + \not p_2}\, \gamma_{\mu_2}\big\}.
\end{aligned} \tag{28}$$

Substituting these equations in eq. (27) and performing appropriate momentum shifts, one easily shows that, in eq. (27), the first integral cancels the fourth integral and the second integral cancels the third one: thus proving that the eq. (25) actually holds. We get finally

$$p_3^{\mu_3}\, \langle j^{a_1\,-}_{\ \mu_1}(p_1)\, j^{a_2\,-}_{\ \mu_2}(p_2)\, j^{a_3\,+}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} = 0; \tag{29}$$

a result which is obtained by substituting eq. (25) in eq. (24). The same conclusion can be reached, using completely analogous methods, for the three-current correlation function $\langle j^{a_1\,+}_{\ \mu_1}(p_1) j^{a_2\,+}_{\ \mu_2}(p_2) j^{a_3\,-}_{\ \mu_3}(p_3)\rangle_{\mathrm{con}}$:

$$p_3^{\mu_3}\, \langle j^{a_1\,+}_{\ \mu_1}(p_1)\, j^{a_2\,+}_{\ \mu_2}(p_2)\, j^{a_3\,-}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} = 0. \tag{30}$$

Things do not work the same way for the remaining Green functions on the r.h.s. of eq. (22). Actually, each three-current correlation function gives a contribution, vanishing the sum of them all. Let us see it. Some algebra leads to

$$\begin{aligned}
p_3^{\mu_3}\, \langle j^{a_1\,-}_{\ \mu_1}(p_1)\, j^{a_2\,+}_{\ \mu_2}(p_2)\, j^{a_3\,-}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} &= (2\pi)^4\delta(p_1+p_2+p_3)\mathrm{Tr}(\mathrm{T}^{a_1}\mathrm{T}^{a_3})\mathrm{Tr}\,\mathrm{T}^{a_2} \\
&\Big[e^{-\frac{i}{2}\theta(p_1,p_2)} \triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_2) + e^{\frac{i}{2}\theta(p_1,p_2)} \triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_2)\Big], \\
p_3^{\mu_3}\, \langle j^{a_1\,+}_{\ \mu_1}(p_1)\, j^{a_2\,-}_{\ \mu_2}(p_2)\, j^{a_3\,+}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} &= -(2\pi)^4\delta(p_1+p_2+p_3)\mathrm{Tr}(\mathrm{T}^{a_1}\mathrm{T}^{a_3})\mathrm{Tr}\,\mathrm{T}^{a_2} \\
&\Big[e^{\frac{i}{2}\theta(p_1,p_2)} \triangle^{(1)+}_{\mu_1\mu_2}(p_1,p_2|\tilde p_2) + e^{-\frac{i}{2}\theta(p_1,p_2)} \triangle^{(2)+}_{\mu_1\mu_2}(p_1,p_2|\tilde p_2)\Big], \\
p_3^{\mu_3}\, \langle j^{a_1\,+}_{\ \mu_1}(p_1)\, j^{a_2\,-}_{\ \mu_2}(p_2)\, j^{a_3\,-}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} &= (2\pi)^4\delta(p_1+p_2+p_3)\mathrm{Tr}(\mathrm{T}^{a_2}\mathrm{T}^{a_3})\mathrm{Tr}\,\mathrm{T}^{a_1} \\
&\Big[e^{-\frac{i}{2}\theta(p_1,p_2)} \triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_1) + e^{\frac{i}{2}\theta(p_1,p_2)} \triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde p_1)\Big], \\
p_3^{\mu_3}\, \langle j^{a_1\,-}_{\ \mu_1}(p_1)\, j^{a_2\,+}_{\ \mu_2}(p_2)\, j^{a_3\,+}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}} &= -(2\pi)^4\delta(p_1+p_2+p_3)\mathrm{Tr}(\mathrm{T}^{a_2}\mathrm{T}^{a_3})\mathrm{Tr}\,\mathrm{T}^{a_1} \\
&\Big[e^{\frac{i}{2}\theta(p_1,p_2)} \triangle^{(1)+}_{\mu_1\mu_2}(p_1,p_2|\tilde p_1) + e^{-\frac{i}{2}\theta(p_1,p_2)} \triangle^{(2)+}_{\mu_1\mu_2}(p_1,p_2|\tilde p_1)\Big].
\end{aligned} \tag{31}$$

In this equation, the contributions denoted by $\triangle^{(1)\pm}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_i)$ and $\triangle^{(2)\pm}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_i)$, with $i=1$ and 2, are given by the following integrals

$$\triangle^{(1)\pm}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_i) \;=\; \int \frac{d^4q}{(2\pi)^4}\, e^{\pm i\theta(q,p_i)}\, \frac{\mathrm{tr}^{\,\mathrm{eps}}\{(\not{q}+\not{p}_1)\,\gamma_{\mu_1}\mathrm{P}_+\,\not{q}\gamma_{\mu_2}\mathrm{P}_+\,(\not{q}-\not{p}_2)(\not{p}_1+\not{p}_2)\mathrm{P}_+\}}{(q^2+i0^+)((q+p_1)^2+i0^+)((q-p_2)^2+i0^+)},$$

and

$$\triangle^{(2)\pm}_{\mu_1\mu_2}(p_1,p_2\tilde{p}_i) \;=\; \int \frac{d^4q}{(2\pi)^4}\, e^{\pm i\theta(q,p_i)}\, \frac{\mathrm{tr}^{\,\mathrm{eps}}\{(\not{q}+\not{p}_2)\,\gamma_{\mu_2}\mathrm{P}_+\,\not{q}\gamma_{\mu_1}\mathrm{P}_+\,(\not{q}-\not{p}_1)(\not{p}_1+\not{p}_2)\mathrm{P}_+\}}{(q^2+i0^+)((q+p_2)^2+i0^+)((q-p_1)^2+i0^+)}.$$

Using the same variety of tricks that led to eq. (25), one shows that now

$$\begin{aligned}
&e^{\mp\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_i)+e^{\pm\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_i)=\\
&\mp 4\sin\frac{1}{2}\theta(p_1,p_2)\,\varepsilon_{\mu_1\mu_2\alpha\beta}\int\frac{d^4q}{(2\pi)^4}\,e^{\mp i\theta(q,p_i)}\,\frac{q^\alpha p_i^\beta}{(q^2+i0^+)((q+p_i)^2+i0^+)},
\end{aligned}\qquad(32)$$

where $i=1$ and 2. For the sake of the reader, we shall spell out the computations leading to the previous equation. Let us change variables $q\to q+p_2$ and $q\to q+p_1$ in $\triangle^{(1)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)$ and $\triangle^{(2)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)$, respectively, and use the cyclicity of the trace, to obtain

$$\begin{aligned}
&e^{\mp\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)+e^{\pm\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)=\\
&e^{\mp\frac{i}{2}\theta(p_1,p_2)}\int\frac{d^4q}{(2\pi)^4}\,e^{\mp i\theta(q,p_2)}\,\mathrm{tr}^{\,\mathrm{eps}}\{\frac{1}{\not{q}}(\not{p}_1+\not{p}_2)\mathrm{P}_+\,\frac{1}{\not{q}+\not{p}_1+\not{p}_2}\,\gamma_{\mu_1}\mathrm{P}_+\,\frac{1}{\not{q}+\not{p}_2}\,\gamma_{\mu_2}\mathrm{P}_+\}+\\
&e^{\mp\frac{i}{2}\theta(p_1,p_2)}\int\frac{d^4q}{(2\pi)^4}\,e^{\mp i\theta(q,p_2)}\,\mathrm{tr}^{\,\mathrm{eps}}\{\frac{1}{\not{q}}(\not{p}_1+\not{p}_2)\mathrm{P}_+\,\frac{1}{\not{q}+\not{p}_1+\not{p}_2}\,\gamma_{\mu_2}\mathrm{P}_+\,\frac{1}{\not{q}+\not{p}_1}\,\gamma_{\mu_1}\mathrm{P}_+\}.
\end{aligned}\qquad(33)$$

Notice that, unlike eq. (26), the exponential factor in front of each integral is the same. This will turn out to be of the utmost importance. Next, let us use the equations $\gamma_\mu\gamma_\nu\mathrm{P}_+=\mathrm{P}_+\gamma_\mu\gamma_\nu$, $\mathrm{P}_+^2=\mathrm{P}_+$ and

$$(\not{p}_1+\not{p}_2)\gamma_5=-\not{q}\gamma_5-\gamma_5(\not{q}+\not{p}_1+\not{p}_2),$$

to cast the r.h.s of eq. (33) into the form

$$\begin{aligned}
&-\frac{1}{2}\,e^{\mp\frac{i}{2}\theta(p_1,p_2)}\int\frac{d^4q}{(2\pi)^4}\,e^{\mp i\theta(q,p_2)}\,\Big[\mathrm{tr}\{\gamma_5\,\frac{1}{\not{q}+\not{p}_1+\not{p}_2}\,\gamma_{\mu_1}\,\frac{1}{\not{q}+\not{p}_2}\,\gamma_{\mu_2}\}+\\
&\qquad\qquad\mathrm{tr}\{\frac{1}{\not{q}}\,\gamma_5\gamma_{\mu_1}\,\frac{1}{\not{q}+\not{p}_2}\,\gamma_{\mu_2}\}\Big]\\
&-\frac{1}{2}e^{\mp\frac{i}{2}\theta(p_1,p_2)}\int\frac{d^4q}{(2\pi)^4}\,e^{\mp i\theta(q,p_2)}\,\Big[\mathrm{tr}\{\gamma_5\,\frac{1}{\not{q}+\not{p}_1+\not{p}_2}\,\gamma_{\mu_2}\,\frac{1}{\not{q}+\not{p}_1}\,\gamma_{\mu_1}\}+\\
&\qquad\qquad\mathrm{tr}\{\frac{1}{\not{q}}\,\gamma_5\gamma_{\mu_2}\,\frac{1}{\not{q}+\not{p}_1}\,\gamma_{\mu_1}\}\Big].
\end{aligned}\qquad(34)$$

Recall that $\mathrm{tr}^{\,\mathrm{eps}}$ means that one only keeps contributions that carry the Levi-Civita symbol. Taking into account eq. (28), one obtains that eq. (34) can be written as follows

$$
\begin{aligned}
-\frac{1}{2}\, e^{\mp\frac{i}{2}\theta(p_1,p_2)}\int \frac{d^4q}{(2\pi)^4}\, e^{\mp i\theta(q,p_2)} \Big[&-\mathrm{tr}\{\frac{1}{\not q+\not p_2}\,\gamma_5\,\gamma_{\mu_2}\,\frac{1}{\not q+\not p_1+\not p_2}\,\gamma_{\mu_1}\}\\
&+\mathrm{tr}\{\frac{1}{\not q}\,\gamma_5\gamma_{\mu_1}\,\frac{1}{\not q+\not p_2}\,\gamma_{\mu_2}\}\Big]\\
-\frac{1}{2}e^{\mp\frac{i}{2}\theta(p_1,p_2)}\int \frac{d^4q}{(2\pi)^4}\, e^{\mp i\theta(q,p_2)} \Big[&-\mathrm{tr}\{\frac{1}{\not q+\not p_1}\,\gamma_5\,\gamma_{\mu_1}\,\frac{1}{\not q+\not p_1+\not p_2}\,\gamma_{\mu_2}\}\\
&+\mathrm{tr}\{\frac{1}{\not q}\,\gamma_5\gamma_{\mu_2}\,\frac{1}{\not q+\not p_1}\,\gamma_{\mu_1}\}\Big].
\end{aligned}
\tag{35}
$$

Let us next make the following shifts, $q \to q - p_2$ and $q \to q - p_1$, in the first and third integrals in eq. (35). Then, we readily see that the first integral cancels the fourth integral of eq. (35), but the sum of the second and third integrals of eq. (35) yield

$$
\frac{1}{2}\left(e^{\pm\frac{i}{2}\theta(p_1,p_2)}-e^{\mp\frac{i}{2}\theta(p_1,p_2)}\right)\int \frac{d^4q}{(2\pi)^4}\, e^{\mp i\theta(q,p_2)}\,\mathrm{tr}\{\frac{1}{\not q}\,\gamma_5\gamma_{\mu_1}\,\frac{1}{\not q+\not p_2}\,\gamma_{\mu_2}\}.
$$

From this equation one obtains eq. (32) for $i = 2$. Let us now replace the integral in eq. (32) with its value, which can be found in the appendix. One obtains, for $i = 2$, that

$$
\begin{aligned}
e^{\mp\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde p_i)+e^{\pm\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)\mp}_{\mu_1\mu_2}(p_1,p_2|\tilde p_i) &= \frac{1}{2\pi^2}\sin\frac{1}{2}\theta(p_1,p_2)\\
\varepsilon_{\mu_1\mu_2\alpha\beta}\,\frac{\tilde p_i^{\alpha}p_i^{\beta}}{\tilde p_i^2}\int_0^1 dx\,\sqrt{\tilde p_i^2(-p_i^2-i0^+)x(1-x)}\;\,\mathrm{K}_1\Big(&\sqrt{\tilde p_i^2(-p_i^2-i0^+)x(1-x)}\Big);
\end{aligned}
\tag{36}
$$

a result which is also valid for $i = 1$. Let us warn the reader that we use the notation $\tilde p_i^{\mu} = \theta^{\mu\nu}p_{i\,\nu}$ and $\tilde p_i^2 \equiv p_{i\,\mu}\,\theta^{\mu\rho}\eta_{\rho\sigma}\theta^{\sigma\nu}\,p_{i\,\nu}$, so that $\tilde p_i^2 \geq 0$. Substituting this result in eqs. (31), one comes to the conclusion that there is a pairwise cancellation mechanism at work:

$$
\begin{aligned}
&p_3^{\mu_3}\,\langle j^{a_1\,-}_{\ \mu_1}(p_1)\,j^{a_2\,+}_{\ \mu_2}(p_2)\,j^{a_3\,-}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}}+p_3^{\mu_3}\,\langle j^{a_1\,+}_{\ \mu_1}(p_1)\,j^{a_2\,-}_{\ \mu_2}(p_2)\,j^{a_3\,+}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}}=0,\\
&p_3^{\mu_3}\,\langle j^{a_1\,+}_{\ \mu_1}(p_1)\,j^{a_2\,-}_{\ \mu_2}(p_2)\,j^{a_3\,-}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}}+p_3^{\mu_3}\,\langle j^{a_1\,-}_{\ \mu_1}(p_1)\,j^{a_2\,+}_{\ \mu_2}(p_2)\,j^{a_3\,+}_{\ \mu_3}(p_3)\rangle^{\mathrm{eps}}_{\mathrm{con}}=0.
\end{aligned}
\tag{37}
$$

Finally, taking into account eqs. (22), (29), (30) and (37), one concludes that in the pseudotensor part of $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)_{\mathrm{NP}}$ no chiral gauge anomaly occurs, i.e.,

$$
p_3^{\mu_3}\,\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2,p_3)^{\mathrm{eps}}_{\mathrm{NP}}=0.
$$

We have thus shown that a noncommutative $U(N)$ chiral theory with only chiral adjoint fermions do not present a chiral anomaly in the three point function (triangle anomaly). The descent equations [5] leads to the conclusion that noncommutative $U(N)$ chiral gauge theory with only adjoint fermions is anomaly free.

Let us now study the gauge anomalies of the theory with action in eq. (12). If this theory were gauge invariant at the quantum level the Ward identities should read thus

$$
\begin{aligned}
\int d^4x\, \omega^{i_1}_{\ i_2} \star \partial_\mu \frac{\delta\Gamma[\mathrm{A},\mathrm{B}]}{\delta \mathrm{A}^{i_1}_{\mu\, i_2}} &= i \int d^4x\, \omega^{i_1}_{\ i_2} \star \left[\mathrm{A}^{i_2}_{\mu\, i_3} \star \frac{\delta\Gamma[\mathrm{A},\mathrm{B}]}{\delta \mathrm{A}^{i_1}_{\mu\, i_3}} - \frac{\delta\Gamma[\mathrm{A},\mathrm{B}]}{\delta \mathrm{A}^{i_3}_{\mu\, i_2}} \star \mathrm{A}^{i_3}_{\mu\, i_1} \right], \\
\int d^4x\, \chi^{j_1}_{\ \ j_2} \star \partial_\mu \frac{\delta\Gamma[\mathrm{A},\mathrm{B}]}{\delta \mathrm{B}^{j_1}_{\mu\, j_2}} &= i \int d^4x\, \chi^{j_1}_{\ \ j_2} \star \left[\mathrm{B}^{j_2}_{\mu\, j_3} \star \frac{\delta\Gamma[\mathrm{A},\mathrm{B}]}{\delta \mathrm{B}^{j_1}_{\mu\, j_3}} - \frac{\delta\Gamma[\mathrm{A},\mathrm{B}]}{\delta \mathrm{B}^{j_3}_{\mu\, j_2}} \star \mathrm{B}^{j_3}_{\mu\, j_1} \right].
\end{aligned}
\tag{38}
$$

Let us introduce the following currents

$$
j^a_{\ \mu}(x) \equiv i \frac{\delta S[\mathrm{A},\mathrm{B}]}{\mathrm{A}^a_\mu(x)} \quad \text{and} \quad j^b_{\ \mu}(x) \equiv i \frac{\delta S[\mathrm{A},\mathrm{B}]}{\mathrm{B}^b_\mu(x)}.
$$

Hence,

$$
j^a_{\ \mu}(x) = -i\, \psi^j_{\ k\,\beta} \star \bar\psi^k_{\ i\,\alpha}(x)\, \mathrm{T}^{a\, i}_{U(N)\ j} \left(\bar\gamma^\mu \mathrm{P}_+\right)_{\alpha\beta} \tag{39}
$$

and

$$
j^b_{\ \mu}(x) = -i\, \bar\psi^k_{\ i\,\alpha} \star \psi^i_{\ j\,\beta}(x) \mathrm{T}^{b\, j}_{U(M)\ k} \left(\bar\gamma^\mu \mathrm{P}_+\right)_{\alpha\beta}, \tag{40}
$$

where $\mathrm{T}^a_{U(N)}$ and $\mathrm{T}^b_{U(M)}$ are the generators of $U(N)$ and $U(M)$ in the fundamental representation, respectively. We shall also need the following nonsinglet currents,

$$
j^{(\mathrm{A})\, i_1}_{\ \mu\ \ i_2}(x) = -i\, \psi^{i_1}_{\ \ j\,\beta} \star \bar\psi^j_{\ i_2\,\alpha}(x) \left(\bar\gamma^\mu \mathrm{P}_+\right)_{\alpha\beta} \tag{41}
$$

and

$$
j^{(\mathrm{B})\, j_1}_{\ \mu\ \ j_2}(x) = -i\, \bar\psi^{j_1}_{\ \ i\,\alpha} \star \psi^i_{\ j_2\,\beta}(x) \left(\bar\gamma^\mu \mathrm{P}_+\right)_{\alpha\beta}, \tag{42}
$$

to express the r.h.s of eq. (38) in terms of correlation functions of currents. Unlike the theories previously studied, now, there are nonvanishing pseudotensor contributions to the two-point part of $\Gamma[\mathrm{A},\mathrm{B}]$. These contributions enter the Ward identities in eq. (38).

We have now the following independent three-current correlation functions

$$
\begin{aligned}
&\langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle_{\rm con},\ \langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle_{\rm con}, \\
&\langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle_{\rm con},\ \langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle_{\rm con}, \\
&\langle j^{a_1}_{\ \ \mu_1}(p_1)\, j^{b_2}_{\ \ \mu_2}(p_2)\, j^{a_3}_{\ \ \mu_3}(p_3)\rangle_{\rm con} \quad \text{and} \quad \langle j^{b_1}_{\ \ \mu_1}(p_1)\, j^{a_2}_{\ \ \mu_2}(p_2)\, j^{b_3}_{\ \ \mu_3}(p_3)\rangle_{\rm con}.
\end{aligned}
\tag{43}
$$

The reader should bear in mind that the indices a_i, $i = 1$, 2 and 3, label currents of the type defined in eq. (39), whereas if a current is of the type given in eq. (40) it carries an index b_i, $i = 1$, 2 and 3. In eq. (43) the first two correlation functions are sums of only planar triangle diagrams and the last four are sums of only nonplanar triangle diagrams.

That there be no breaking of the classical gauge symmetry of the theory at hand in the triangle diagrams, demands that the following equations hold

$$
\begin{aligned}
&p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = 0,\\
&p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = 0,\\
&p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = 0,\\
&p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = 0,\\
&p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = -\,e^{\frac{i}{2}\theta(p_1,p_2)}\,{\rm T}^{a_1\,i_1}{}_{i_3}\,{\rm T}^{a_3\,i_3}{}_{i_2}\,\langle j^{({\rm A})\,i_2}{}_{\mu_1\,i_1}(-p_2)\, j^{b_2}{}_{\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con}\\
&\qquad\qquad\qquad + e^{-\frac{i}{2}\theta(p_1,p_2)}{\rm T}^{a_3\,i_1}{}_{i_3}\,{\rm T}^{a_1\,i_3}{}_{i_2}\,\langle j^{({\rm A})\,i_2}{}_{\mu_1\,i_1}(-p_2)\, j^{b_2}{}_{\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con},\\
&p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = -\,e^{\frac{i}{2}\theta(p_1,p_2)}\,{\rm T}^{b_1\,j_1}{}_{j_3}\,{\rm T}^{b_3\,j_3}{}_{j_2}\,\langle j^{({\rm B})\,j_2}{}_{\mu_1\,j_1}(-p_2)\, j^{a_2}{}_{\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con}\\
&\qquad\qquad\qquad + e^{-\frac{i}{2}\theta(p_1,p_2)}{\rm T}^{b_3\,j_1}{}_{j_3}\,{\rm T}^{b_1\,j_3}{}_{j_2}\,\langle j^{({\rm B})\,j_2}{}_{\mu_1\,j_1}(-p_2)\, j^{a_2}{}_{\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con}.
\end{aligned}
\tag{44}
$$

Where $p_3=-p_1-p_2$. The nonsinglet currents $j^{({\rm A})}_{\mu}$ and $j^{({\rm B})}_{\mu}$ are defined in eqs. (41) and (42), respectively. To obtain the previous equation, we have taken into account eq. (38) and the result that the only two-point contribution to $\Gamma[{\rm A},{\rm B}]$ which carries a pseudontensor contribution is of the type

$$
\int d^4x\int d^4y\ {\rm Tr}\,{\rm A}_{\mu_1}(x)\ {\rm Tr}\,{\rm B}_{\mu_2}(y)\ {\rm f}^{\mu_1\mu_2}\,(x,y|\theta),
$$

with

$$
\begin{aligned}
{\rm f}^{\mu_1\mu_2}\,(x,y|\theta) &= \int \frac{d^4p}{(2\pi)^4}\,e^{-ip(x-y)}\ {\rm f}^{\mu_1\mu_2}\,(p|\tilde p),\\
{\rm f}^{\mu_1\mu_2}\,(p|\tilde p) &= \frac{i}{4\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,\frac{\tilde p^\alpha p^\beta}{\tilde p^2}\\
&\quad \int_0^1 dx\,\sqrt{\tilde p^2(-p^2-i0^+)x(1-x)}\ \ {\rm K}_1\Big(\sqrt{\tilde p^2(-p^2-i0^+)x(1-x)}\Big).
\end{aligned}
$$

This pseudotensor contribution is nonplanar and causes no anomaly.

Let us see that the first two identities in eq. (44) do not hold, so that they are anomalous, but the all the others do. The computations we have carried out for the theory with adjoint fermion fields can be readily adapted to the case at hand to obtain

$$
\begin{aligned}
&p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\, j^{a_2}{}_{\mu_2}(p_2)\, j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = -(2\pi)^4\delta(p_1+p_2+p_3)\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta\\
&M\Big(\,{\rm Tr}\,\{{\rm T}^{a_1}_{U(N)},{\rm T}^{a_2}_{U(N)}\}\,{\rm T}^{a_3}_{U(N)}\cos\frac{1}{2}\theta(p_1,p_2) - i{\rm Tr}\,[{\rm T}^{a_1}_{U(N)},{\rm T}^{a_2}_{U(N)}]{\rm T}^{a_3}_{U(N)}\sin\frac{1}{2}\theta(p_1,p_2)\Big),\\
&p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\, j^{b_2}{}_{\mu_2}(p_2)\, j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = (2\pi)^4\delta(p_1+p_2+p_3)\frac{1}{24\pi^2}\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta\\
&N\Big(\,{\rm Tr}\,\{{\rm T}^{b_1}_{U(M)},{\rm T}^{b_2}_{U(M)}\}\,{\rm T}^{b_3}_{U(M)}\cos\frac{1}{2}\theta(p_1,p_2) - i{\rm Tr}\,[{\rm T}^{b_1}_{U(M)},{\rm T}^{b_2}_{U(M)}]{\rm T}^{b_3}_{U(M)}\sin\frac{1}{2}\theta(p_1,p_2)\Big)
\end{aligned}
\tag{45}
$$

and

$$p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\,j^{b_2}{}_{\mu_2}(p_2)\,j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = -(2\pi)^4\delta(p_1+p_2+p_3){\rm Tr}({\rm T}^{b_1}_{U(M)}{\rm T}^{b_2}_{U(M)}){\rm Tr}\,{\rm T}^{a_3}_{U(N)}$$
$$\Big[e^{-\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)+}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)+e^{\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)+}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)\Big],$$
$$p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\,j^{a_2}{}_{\mu_2}(p_2)\,j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = (2\pi)^4\delta(p_1+p_2+p_3){\rm Tr}({\rm T}^{a_1}_{U(N)}{\rm T}^{a_2}_{U(N)}){\rm Tr}\,{\rm T}^{b_3}_{U(M)}$$
$$\Big[e^{\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)+e^{-\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_3)\Big],$$
$$p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\,j^{b_2}{}_{\mu_2}(p_2)\,j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = (2\pi)^4\delta(p_1+p_2+p_3){\rm Tr}({\rm T}^{a_1}_{U(N)}{\rm T}^{a_3}_{U(N)}){\rm Tr}\,{\rm T}^{b_2}_{U(M)}$$
$$\Big[e^{-\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)+e^{\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)-}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)\Big],$$
$$p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\,j^{a_2}{}_{\mu_2}(p_2)\,j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = -(2\pi)^4\delta(p_1+p_2+p_3){\rm Tr}({\rm T}^{b_1}_{U(M)}{\rm T}^{b_3}_{U(M)}){\rm Tr}\,{\rm T}^{a_2}_{U(N)}$$
$$\Big[e^{\frac{i}{2}\theta(p_1,p_2)}\triangle^{(1)+}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)+e^{-\frac{i}{2}\theta(p_1,p_2)}\triangle^{(2)+}_{\mu_1\mu_2}(p_1,p_2|\tilde{p}_2)\Big]. \tag{46}$$

From eq. (45), one deduces that the anomaly cancellation condition for the planar triangle diagrams reads

$${\rm Tr}({\rm T}^{a_1}_{U(N)}{\rm T}^{a_2}_{U(N)}{\rm T}^{a_3}_{U(N)})=0 \qquad \text{and} \qquad {\rm Tr}({\rm T}^{b_1}_{U(M)}{\rm T}^{b_2}_{U(M)}{\rm T}^{b_3}_{U(M)})=0.$$

Both the anomalies which gives rise to these anomaly cancellation conditions are analogous to the anomaly in eq. (10), i.e., the anomaly for chiral fundamental fermions. If we now substitute eq. (36) in eq (46), we shall conclude that the left hand sides of last two identities in eq. (44) do not vanish, but read, respectively, thus

$$p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\,j^{b_2}{}_{\mu_2}(p_2)\,j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = (2\pi)^4\delta(p_1+p_2+p_3){\rm Tr}({\rm T}^{a_1}_{U(N)}{\rm T}^{a_3}_{U(N)}){\rm Tr}\,{\rm T}^{b_2}_{U(M)}$$
$$\Big[\frac{1}{2\pi^2}\sin\frac{1}{2}\theta(p_1,p_2)\;\varepsilon_{\mu_1\mu_2\alpha\beta}\,\frac{\tilde{p}_2^\alpha p_2^\beta}{\tilde{p}_2^2}$$
$$\int_0^1 dx\,\sqrt{\tilde{p}_2^2(-p_2^2-i0^+)x(1-x)}\;\;{\rm K}_1\Big(\sqrt{\tilde{p}_2^2(-p_2^2-i0^+)x(1-x)}\Big)\Big],$$
$$p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\,j^{a_2}{}_{\mu_2}(p_2)\,j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con} = -(2\pi)^4\delta(p_1+p_2+p_3){\rm Tr}({\rm T}^{b_1}_{U(M)}{\rm T}^{b_3}_{U(M)}){\rm Tr}\,{\rm T}^{a_2}_{U(N)}$$
$$\Big[\frac{1}{2\pi^2}\sin\frac{1}{2}\theta(p_1,p_2)\;\varepsilon_{\mu_1\mu_2\alpha\beta}\,\frac{\tilde{p}_2^\alpha p_2^\beta}{\tilde{p}_2^2}$$
$$\int_0^1 dx\,\sqrt{\tilde{p}_2^2(-p_2^2-i0^+)x(1-x)}\;\;{\rm K}_1\Big(\sqrt{\tilde{p}_2^2(-p_2^2-i0^+)x(1-x)}\Big)\Big]. \tag{47}$$

Recall that $\tilde{p}_2^\mu=\theta^{\mu\nu}p_{2\,\nu}$ and $\tilde{p}_2^2\equiv p_{2\,\mu}\,\theta^{\mu\rho}\eta_{\rho\sigma}\theta^{\sigma\nu}\,p_{2\,\nu}$, so that $\tilde{p}_2^2\geq 0$.

Finally, eqs. (25) imply that

$$p_3^{\mu_3}\,\langle j^{a_1}{}_{\mu_1}(p_1)\,j^{a_2}{}_{\mu_2}(p_2)\,j^{b_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con}=0.$$

Similarly,

$$p_3^{\mu_3}\,\langle j^{b_1}{}_{\mu_1}(p_1)\,j^{b_2}{}_{\mu_2}(p_2)\,j^{a_3}{}_{\mu_3}(p_3)\rangle^{\rm eps}_{\rm con}=0.$$

To show that indeed the last four identities in eq. (44) hold, all that remains for us to do is to work out the following expressions

$$-e^{\frac{i}{2}\theta(p_1,p_2)}\,\mathrm{T}^{a_1\,i_1}_{\;\;\;i_3}\,\mathrm{T}^{a_3\,i_3}_{\;\;\;i_2}\,\langle j^{(\mathrm{A})\,i_2}_{\;\;\mu_1\,i_1}(-p_2)\,j^{b_2}_{\;\;\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con}$$
$$+\,e^{-\frac{i}{2}\theta(p_1,p_2)}\mathrm{T}^{a_3\,j_1}_{\;\;\;j_3}\,\mathrm{T}^{a_1\,j_3}_{\;\;\;j_2}\,\langle j^{(\mathrm{A})\,j_2}_{\;\;\mu_1\,j_1}(-p_2)\,j^{b_2}_{\;\;\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con},$$
$$-e^{\frac{i}{2}\theta(p_1,p_2)}\,\mathrm{T}^{b_1\,j_1}_{\;\;\;j_3}\,\mathrm{T}^{b_3\,j_3}_{\;\;\;j_2}\,\langle j^{(\mathrm{B})\,j_2}_{\;\;\mu_1\,j_1}(-p_2)\,j^{a_2}_{\;\;\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con}$$
$$+\,e^{-\frac{i}{2}\theta(p_1,p_2)}\mathrm{T}^{b_3\,j_1}_{\;\;\;j_3}\,\mathrm{T}^{b_1\,j_3}_{\;\;\;j_2}\,\langle j^{(\mathrm{B})\,j_2}_{\;\;\mu_1\,j_1}(-p_2)\,j^{a_2}_{\;\;\mu_2}(p_2)\,\rangle^{\rm eps}_{\rm con}.$$

It is not difficult to see that the previous expressions are equal to

$$-\,i\,\mathrm{Tr}(\mathrm{T}^{a_1}_{U(N)}\mathrm{T}^{a_3}_{U(N)})\mathrm{Tr}\,\mathrm{T}^{b_2}_{U(M)}\;\sin\frac{1}{2}\theta(p_1,p_2)\;\int\frac{dq^4}{(2\pi)^2}\,e^{-i\theta(q,p_2)}\,\frac{\mathrm{tr}\{(\not q+\not p_2)\gamma_{\mu_2}\not q\gamma_{\mu_1}\gamma_5\}}{(q^2+i0^+)((q+p_2)^2+i0^+)},$$
$$-\,i\mathrm{Tr}(\mathrm{T}^{b_1}_{U(M)}\mathrm{T}^{b_3}_{U(M)})\mathrm{Tr}\,\mathrm{T}^{a_2}_{U(N)}\;\sin\frac{1}{2}\theta(p_1,p_2)\;\int\frac{dq^4}{(2\pi)^2}\,e^{i\theta(q,p_2)}\,\frac{\mathrm{tr}\{(\not q+\not p_2)\gamma_{\mu_2}\not q\gamma_{\mu_1}\gamma_5\}}{(q^2+i0^+)((q+p_2)^2+i0^+)},$$

respectively. Some algebra and the help of the appendix makes it possible for us to conclude that the right hand sides of the last two identities in eq. (44) agree, respectively, with their left hand sides, the latter being given in eq. (47).

In summary, we have shown that the last four identities of eq. (44) indeed hold in the quantum theory. These identities are the Ward identities for the nonplanar contributions to the three-point function of $\Gamma[\mathrm{A},\mathrm{B}]$: the Ward identities for the nonplanar triangle contributions. Hence, the nonplanar triangle contributions give rise to no gauge anomaly. On the other hand, the planar triangle contributions are anomalous with anomalies given in eq. (45).

3. The IR origin of nonabelian chiral gauge anomalies

In the previous section we have shown that, for the theories we are discussing, only planar triangle diagrams gives rise to a gauge anomaly and we have given an UV interpretation of this anomaly. Eq. (10) is the basic building-block for this type of anomaly: see eq. (45). To interpret the nonabelian chiral anomaly under scrutiny as an IR phenomenon, we shall follow Coleman and Grossman [7] and compute $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}$ at the point

$$p_1^2=p_2^2=p_3^2=-Q^2,\;p_1+p_2+p_3=0.$$

$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}$ is the pseudotensor part of the three-point function for a noncommutative gauge theory with a right-handed fundamental fermion. The action of this theory is given in eq. (5). The corresponding IR analysis for the planar triangle diagrams arising in the other theories studied in this paper (see eqs. (6), (7) and (12)) can be readily done by adapting the results presented in the sequel.

Let us recall first that formally $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}$ is given by the sum of the pseudotensor contributions coming from the triangle diagrams in fig. 2, which for the case at hand reads

$$\begin{aligned}\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\rm eps}=&e^{-\frac{i}{2}\theta(p_1,p_2)}\,\mathrm{Tr}\,\mathrm{T}^{a_1}\mathrm{T}^{a_2}\mathrm{T}^{a_3}\,\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)+\\&e^{\frac{i}{2}\theta(p_1,p_2)}\,\mathrm{Tr}\,\mathrm{T}^{a_2}\mathrm{T}^{a_1}\mathrm{T}^{a_3}\,\triangle^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2),\end{aligned}\tag{48}$$

where

$$\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2) \;=\; \int \frac{d^4q}{(2\pi)^4}\,\frac{\mathrm{tr}^{\,\mathrm{eps}}\big\{(\not q+\not p_1)\,\gamma_{\mu_1}\mathrm{P}_+\,\not q\gamma_{\mu_2}\mathrm{P}_+\,(\not q-\not p_2)\,\gamma_{\mu_3}\mathrm{P}_+\big\}}{(q^2+i0^+)((q+p_1)^2+i0^+)((q-p_2)^2+i0^+)},$$

and

$$\triangle^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2) \;=\; \int \frac{d^4q}{(2\pi)^4}\,\frac{\mathrm{tr}^{\,\mathrm{eps}}\big\{(\not q+\not p_2)\,\gamma_{\mu_2}\mathrm{P}_+\,\not q\gamma_{\mu_1}\mathrm{P}_+\,(\not q-\not p_1)\,\gamma_{\mu_3}\mathrm{P}_+\big\}}{(q^2+i0^+)((q+p_2)^2+i0^+)((q-p_1)^2+i0^+)}.$$

The symbol $\mathrm{tr}^{\,\mathrm{eps}}$ denotes the pseudotensor contributions, i.e., contributions involving an odd number of γ_5 matrices.

As they stand the Feynman amplitudes $\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ and $\triangle^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ above are at first sight formal expressions since they are sum of UV divergent by power-counting Feynman integrals. However, we shall see in a moment that one can associate to these Feynman amplitudes a unique tempered distribution provided cyclicity of the external indices and momenta is imposed. Indeed, renormalization theory [14] associates to every formal Feynman amplitude a tempered distribution which is uniquely defined up to a local polynomial of the appropriate dimension in the external momenta*. This polynomial can be further restricted by symmetries. Hence, the Feynman amplitude $\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ can be uniquely defined as a distribution modulo the following polynomial

$$C_1\,\varepsilon_{\mu_1\mu_2\mu_3\alpha}\,p_1^{\alpha} \;+\; C_2\,\varepsilon_{\mu_1\mu_2\mu_3\alpha}\,p_2^{\alpha}, \tag{49}$$

where C_1 and C_2 are arbitrary constants. If we next impose symmetry under cyclic permutations of the pairs $(\mu_1,p_1),(\mu_2,p_2),(\mu_3,p_3)$, with $p_1+p_2+p_3=0$, then C_1 and C_2 are fixed for once and all. Indeed, any further addition ought to be of the type

$$C_3\,\varepsilon_{\mu_1\mu_2\mu_3\alpha}\,(p_1\;+\;p_2\;+\;p_3)^{\alpha},$$

which vanishes upon imposing four-momentum conservation. Actually, what this discussion is telling us is that if we use, as intermediate computational procedure, a regularization method that explicitly preserves the formal symmetry of $\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ under cyclic permutations of the pairs (μ_1,p_1), (μ_2,p_2), (μ_3,p_3), the limit in which the regulator is removed is well-defined. Besides, this limit is the same for all regularizations (and, of course, renormalizations) of $\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ which preserve its formal cyclic symmetry. Of course, any renormalization which breaks this cyclic symmetry can be brought to the unique symmetric form just mentioned by adding a finite counterterm of the form given in eq. (49). It is in this sense that we are entitled to say that the Feynman amplitude $\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ is an UV finite quantity, in spite of the fact that it is not UV finite by power-counting. The same kind of arguments can be applied to $\triangle^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ to conclude that it is also an UV finite object, though it is not UV finite by power-counting.

* Here we assume that, since the diagrams we are considering are one-loop and planar, standard renormalization theory can be applied to each diagram without further ado.

There is a very handy regularization procedure which explicitly preserves the symmetry of each triangle diagram in fig. 2 under cyclic permutations of its external legs. This is the dimensional regularization algorithm set up in the previous section. The dimensionally regularized counterparts of $\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ and $\triangle^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$ read:

$$\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2;d) \;=\; \int \frac{d^dq}{(2\pi)^d}\,\frac{\mathrm{tr}^{\,\mathrm{eps}}\big\{(\not q+\not p_1)\,\bar\gamma_{\mu_1}\mathrm{P}_+\,\not q\bar\gamma_{\mu_2}\mathrm{P}_+\,(\not q-\not p_2)\,\bar\gamma_{\mu_3}\mathrm{P}_+\big\}}{(q^2+i0^+)((q+p_1)^2+i0^+)((q-p_2)^2+i0^+)},$$

and

$$\triangle^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2;d) \;=\; \int \frac{d^dq}{(2\pi)^d}\,\frac{\mathrm{tr}^{\,\mathrm{eps}}\big\{(\not q+\not p_2)\,\bar\gamma_{\mu_2}\mathrm{P}_+\,\not q\bar\gamma_{\mu_1}\mathrm{P}_+\,(\not q-\not p_1)\,\bar\gamma_{\mu_3}\mathrm{P}_+\big\}}{(q^2+i0^+)((q+p_2)^2+i0^+)((q-p_1)^2+i0^+)}. \tag{50}$$

Taking into account that

$$\mathrm{P}_+\hat\gamma_\mu\bar\gamma_\nu\mathrm{P}_+=0,\;\; \mathrm{P}_+\bar\gamma_\mu\bar\gamma_\nu=\bar\gamma_\mu\bar\gamma_\nu\mathrm{P}_+,$$

we conclude that eq. (50) can be turned into the following one

$$\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2;d) \;=\; \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\,\frac{\mathrm{tr}\big\{(\bar{\not q}+\bar{\not p}_1)\,\bar\gamma_{\mu_1}\,\bar{\not q}\bar\gamma_{\mu_2}\,(\bar{\not q}-\bar{\not p}_2)\,\bar\gamma_{\mu_3}\gamma_5\big\}}{(q^2+i0^+)((q+p_1)^2+i0^+)((q-p_2)^2+i0^+)},$$

and

$$\triangle^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2;d) \;=\; \frac{1}{2}\int \frac{d^dq}{(2\pi)^d}\,\frac{\mathrm{tr}\big\{(\bar{\not q}+\bar{\not p}_2)\,\bar\gamma_{\mu_2}\,\bar{\not q}\bar\gamma_{\mu_1}\,(\bar{\not q}-\bar{\not p}_1)\,\bar\gamma_{\mu_3}\gamma_5\big\}}{(q^2+i0^+)((q+p_2)^2+i0^+)((q-p_1)^2+i0^+)}. \tag{51}$$

The computation of the previous integrals at $p_1^2=p_2^2=p_3^2=-Q^2$ is very easy. The substitution in eq. (51) of the integrals in the appendix and some self-evident algebraic arrangements yield upon taking the limit $d\to 4$ the following result

$$\triangle^{(1)}_{\mu_1\mu_2\mu_3}(p_1,p_2) \;=\; \triangle^{(2)}_{\mu_1\mu_2\mu_3}(p_1,p_2)$$
$$=\frac{1}{24\pi^2}\left(\frac{1}{Q^2}\right)\big(\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta\,p_{3\,\mu_3}+\varepsilon_{\mu_3\mu_1\alpha\beta}\,p_3^\alpha p_1^\beta\,p_{2\,\mu_2}+\varepsilon_{\mu_2\mu_3\alpha\beta}\,p_2^\alpha p_3^\beta\,p_{1\,\mu_1}\big). \tag{52}$$

The Feynman amplitudes in the previous equation have poles at $Q^2=0$ and they are these IR singularities which we shall hold responsible for the existence of the nonabelian chiral anomaly [7]. If we now substitute eq. (52) into eq. (48) we will obtain the whole anomalous contribution to the three-point function at $p_1^2=p_2^2=p_3^2=-Q^2$:

$$\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\mathrm{eps}} \;=$$
$$\frac{1}{24\pi^2}\left(\frac{1}{Q^2}\right)\left(\mathrm{Tr}\,\{\mathrm{T}^{a_1},\mathrm{T}^{a_2}\}\,\mathrm{T}^{a_3}\cos\frac{1}{2}\theta(p_1,p_2)-i\mathrm{Tr}\,[\mathrm{T}^{a_1},\mathrm{T}^{a_2}]\mathrm{T}^{a_3}\sin\frac{1}{2}\theta(p_1,p_2)\right)$$
$$\big(\,\varepsilon_{\mu_1\mu_2\alpha\beta}\,p_1^\alpha p_2^\beta\,p_{3\,\mu_3}+\varepsilon_{\mu_3\mu_1\alpha\beta}\,p_3^\alpha p_1^\beta\,p_{2\,\mu_2}+\varepsilon_{\mu_2\mu_3\alpha\beta}\,p_2^\alpha p_3^\beta\,p_{1\,\mu_1}\big). \tag{53}$$

Notice that by contracting with $p_3^{\mu_3}$ both sides of the previous equation, one obtains once again the anomaly equation (eq. (10)). Also notice that unlike in the commutative case the r.h.s of eq. (53) vanishes if and only if $\mathrm{Tr}\,\{\mathrm{T}^{a_1},\mathrm{T}^{a_2}\}\,\mathrm{T}^{a_3}=0$ and $\mathrm{Tr}\,[\mathrm{T}^{a_1},\mathrm{T}^{a_2}]\mathrm{T}^{a_3}=0$, i.e., $\mathrm{Tr}\;\mathrm{T}^{a_1}\mathrm{T}^{a_2}\mathrm{T}^{a_3}=0$. Indeed, the nonpolynomial -in the Moyal product- IR contributions,

$$\frac{\cos\frac{1}{2}\theta(p_1,p_2)}{Q^2}\quad\text{and}\quad\frac{\sin\frac{1}{2}\theta(p_1,p_2)}{Q^2},$$

in this equation, makes it impossible for us to redefine $\Gamma^{a_1a_2a_3}_{\mu_1\mu_2\mu_3}(p_1,p_2)^{\mathrm{eps}}$ so that the anomaly cancellation condition read merely $\mathrm{Tr}\,\{\mathrm{T}^{a_1},\mathrm{T}^{a_2}\}\,\mathrm{T}^{a_3}=0$.

4. Summary and Conclusions

In this paper we have shown that the one-loop noncommutative nonabelian gauge anomalies for $U(N)$ groups can be interpreted either as an UV effect or as an IR phenomenon. We have considered three basic types of noncommutative chiral gauge theories, namely, gauge theories with a fundamental, gauge theories with an adjoint and gauge theories with a bi-fundamental right-handed fermion. We have computed the anomaly in one-loop planar triangle diagrams and shown that the nonplanar contributions yield no gauge anomaly since they preserve the corresponding Ward identities. It turned out that chiral gauge theories with fundamental, anti-fundamental and bi-fundamental matter are, in general, anomalous and that chiral theories with only adjoint fermions are -due to a special cancellation mechanism- always anomaly free. Last but not least, we have clarified the origin of the noncommutative anomaly cancellation condition $\mathrm{Tr}\,\mathrm{T}^{a_1}\mathrm{T}^{a_2}\mathrm{T}^{a_3}=0$.

It will be interesting to carry out the analysis presented here for the theories introduced in ref. [15] and for the axial anomaly [16]. Anomalies in the presence of noncommutative gravity [17] are also worth studying. We shall report on these topics elsewhere.

Appendix

The following result is needed to obtain eq. (36):

$$\int \frac{d^4q}{(2\pi)^4}\, e^{\pm i\theta(q,p)}\, \frac{q^\mu}{(q^2+i0^+)((q+p)^2+i0^+)} =$$
$$-\frac{i\,p^\mu}{8\pi^2}\int_0^1 dx\, x\, \mathrm{K}_0\Big(\sqrt{\tilde{p}^2(-p^2-i0^+)x(1-x)}\Big)$$
$$\pm\frac{1}{8\pi^2}\frac{\tilde{p}^\mu}{\tilde{p}^2}\int_0^1 dx\,\sqrt{\tilde{p}^2(-p^2-i0^+)x(1-x)}\;\mathrm{K}_1\Big(\sqrt{\tilde{p}^2(-p^2-i0^+)x(1-x)}\Big),$$

where $\tilde{p}^\mu=\theta^{\mu\nu}p_\nu$, but $\tilde{p}^2\equiv p_\mu\,\theta^{\mu\rho}\eta_{\rho\sigma}\theta^{\sigma\nu}\,p_\nu$, so that $\tilde{p}_i^2\geq 0$.

Next, we display the integrals needed to obtain eq. (52). These integrals are worked out at the point $p_1^2=p_2^2=-2\,p_1\cdot p_2=-\,Q^2$. They read

$$\int\frac{d^dq}{(2\pi)^d}\;\frac{1}{q^2\,(q+p_1)^2\,(q-p_2)^2}=\frac{\Phi}{Q^2}\,+\,O(d-4),$$
$$\int\frac{d^dq}{(2\pi)^d}\;\frac{\bar{q}_\mu}{q^2\,(q+p_1)^2\,(q-p_2)^2}=\Big(\frac{\Phi}{3}\Big)\Big(\frac{1}{Q^2}\Big)(\bar{p}_2-\bar{p}_1)_\mu\,+\,O(d-4),$$
$$\int\frac{d^dq}{(2\pi)^d}\;\frac{\bar{q}_\mu\bar{q}_\nu}{q^2\,(q+p_1)^2\,(q-p_2)^2}=\Big(\frac{\mathrm{I}_1}{4}+\frac{\Phi}{6}+\frac{\mathrm{I}_2}{4}\Big)\bar{g}_{\mu\nu}+\frac{\mathrm{I}_2}{6}\Big(\frac{1}{Q^2}\Big)(\bar{p}_{1\,\mu}\bar{p}_{2\,\nu}+\bar{p}_{2\,\mu}\bar{p}_{1\,\nu})+$$
$$\Big(\frac{\Phi}{3}+\frac{\mathrm{I}_2}{3}\Big)\Big(\frac{1}{Q^2}\Big)(\bar{p}_{1\,\mu}\bar{p}_{1\,\nu}+\bar{p}_{2\,\mu}\bar{p}_{2\,\nu})\,+\,O(d-4),$$
$$\int\frac{d^dq}{(2\pi)^d}\;\frac{\bar{q}^2\,\bar{q}_\mu}{q^2\,(q+p_1)^2\,(q-p_2)^2}=\Big(\frac{\mathrm{I}_1}{2}+\frac{\mathrm{I}_2}{6}\Big)(\bar{p}_2-\bar{p}_1)_\mu\,+\,O(d-4),$$

where

$$
\begin{aligned}
\mathrm{I}_1 &= \frac{i}{16\pi^2}\left(-\frac{1}{\epsilon}-\gamma-\ln\,\frac{Q^2}{4\pi\kappa^2}+2\right),\\
\mathrm{I}_2 &= \frac{i}{16\pi^2},\\
\Phi &= \frac{i}{16\pi^2}\int_0^1 dx\,\frac{\ln x(1-x)}{1-x+x^2},
\end{aligned}
$$

and $d = 4+2\epsilon$.

Note that all the ugly features of the integrals above nicely cancel against one another when substituted in eq. (51) to yield the beautiful result of eq. (52).

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